BATALIN VILKOVISKY ALGEBRA STRUCTURES ON THE HOCHSCHILD COHOMOLOGY OF SELF-INJECTIVE NAKAYAMA ALGEBRAS

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Abstract. In this paper, we determine a Batalin Vilkovisky algebra structure on the Hochschild cohomology of self-injective Nakayama algebras over an algebraically closed field.

1. Introduction

Hochschild cohomology is an invariant of derived equivalence and it has several algebraic structures; module structures, graded commutative ring structures and Gerstenhaber algebra structures, etc. These algebraic structures of Hochschild cohomology of algebras have been computed for many classes of algebras.

Tradler [3] discovered that Hochschild cohomology of arbitrary symmetric algebra has a Batalin Vilkovisky (BV) algebra structure given by a symmetric bilinear form. Later, Lambre, Zhou and Zimmermann [2] discovered that Hochschild cohomology of Frobenius algebras with diagonalizable Nakayama automorphism has a BV algebra structure. However, it is not known if Hochschild cohomology of Frobenius algebras has a BV algebra structure in general.

Recently, for any Frobenius algebra A, Volkov [4] defined the cohomology $\mathrm{HH}^*(A)^{\nu\uparrow}$ of Hochschild complex related to Nakayama automorphism ν, which induces Gerstenhaber algebra $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$. Moreover, Volkov also found a BV algebra structure on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$. In particular, if the Nakayama automorphism ν is diagonalizable, then $\mathrm{HH}^*(A)^{\nu\uparrow} \cong \mathrm{HH}^*(A)$ and the BV differential on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$ induces the one on the Gerstenhaber algebra $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$. In [4], the BV differentials on Hochschild cohomology of finite-representation self-injective algebras of tree type $D_n (n \geq 4)$ with diagonalizable Nakayama automorphism have been calculated. However, there are few examples of complete calculation of BV differentials on Hochschild cohomology of Frobenius algebras which are not symmetric.

In this paper, for self-injective Nakayama algebra Λ we determine the cohomology ring $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ and BV differentials on $(\mathrm{HH}^*(\Lambda)^{\nu\uparrow}, \smile, [\ ,\ ])$. Consequently, we have $\mathrm{HH}^*(\Lambda)^{\nu\uparrow} \cong \mathrm{HH}^*(\Lambda)$ as graded commutative algebras and $[\ ,\ ] = 0$ on $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$.

2. Preliminaries

In this section, we recall the definition for Gerstenhaber algebras, Batalin-Vilkovisky algebras, Hochchild cohomology. Moreover, following [4], for a Frobenius algebra A we recall BV differentials on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$

The detailed version of this paper will be submitted for publication elsewhere.

First, we recall the definition of the Gerstenhaber algebras and Batalin Vilkovisky algebras.

Definition 1. A *Gerstenhaber algebra* over an algebraically closed field K is $(V^*, \smile, [\ ,\ ])$, where $V^* = \oplus_{n\geq 0} V^n$ is a graded K-vector space, $\smile: V^n \times V^m \to V^{n+m}$ is a cup product of degree zero and $[\ ,\ ] : V^n \times V^m \to V^{n+m-1}$ is a Lie bracket of degree -1 such that

(i) $(V^*, \smile)$ is a graded commutative associative algebra with unit $1 \in V^0$.
(ii) $(V^*, \smile, [\ ,\ ])$ is a graded Lie algebra.
(iii) For arbitrary homogeneous elements a, b, c in V^*

$$[a, b \smile c] = [a, b] \smile c + (-1)^{(|a|-1)|b|} b \smile [a, c],$$

where the notation $|a|$ means the degree of the homogeneous element a.

Definition 2. A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra $(V^*, \smile, [\ ,\ ])$ with an operator $\Delta : V^* \to V^{*-1}$ of degree -1 such that $\Delta \circ \Delta = 0$ and

$$[a, b] = -(-1)^{(|a|-1)|b|}(\Delta(a \smile b) - \Delta(a) \smile b - (-1)^{|a|} a \smile \Delta(b))$$

for homogeneous elements $a, b \in V^*$.

Let K be an algebraically closed field and A a finite dimensional K-algebra.

Definition 3. The following complex $(C^*(A), \delta_*)$ is called the *Hochschild complex* of A:

$$0 \to C^0(A) \xrightarrow{\delta_0} C^1(A) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-2}} C^{n-1}(A) \xrightarrow{\delta_{n-1}} C^n(A) \xrightarrow{\delta_n} C^{n+1}(A) \xrightarrow{\delta_{n+1}} \cdots$$

where

$$\begin{aligned}
&C^0(A) = \mathrm{Hom}_K(K, A) \cong A\\
&C^n(A) = \mathrm{Hom}_K(A^{\otimes n}, A)\\
&\delta_n(f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1})\\
&\quad + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}
\end{aligned}$$

for $f \in C^n(A)$ and $n \geq 1$. The n-th *Hochschild cohomology group* $\mathrm{HH}^n(A)$ is defined as the n-th cohomology of $(C^*(A), \delta_*)$.

The bar resolution $(\mathrm{Bar}_*(A), b_*)$ of A is the following:

$$\cdots \xrightarrow{b_{n+1}} \mathrm{Bar}_n(A) \xrightarrow{b_n} \mathrm{Bar}_{n-1}(A) \xrightarrow{b_{n-1}} \cdots \xrightarrow{b_2} \mathrm{Bar}_1(A) \xrightarrow{b_1} \mathrm{Bar}_0(A) \xrightarrow{b_0} A \to 0$$

where

$$\begin{aligned}
&\mathrm{Bar}_n(A) = A^{\otimes n+2}\\
&b_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}
\end{aligned}$$

The isomorphism $\mathrm{Hom}_K(A^{\otimes n}, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes n+2}, A)$ for $n \geq 0$ induces an isomorphism $(\mathrm{Hom}_{A^e}(\mathrm{Bar}_*(A), A), \mathrm{Hom}_{A^e}(b_*, A)) \cong (C^*(A), \delta_*)$. Thus, $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$. The cup product on the Hochschild complex $(C^*(A), \delta_*)$ is given as follows: for $f \in C^n(A)$ and $g \in C^m(A)$, $f \smile g \in C^{m+n}(A)$ is given by

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n) g(a_{n+1} \otimes \cdots \otimes a_{n+m})$$

The cup product induces the cup product $\smile : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m}(A)$. Then, $\mathrm{HH}^*(A) := \oplus_{n \geq 0} \mathrm{HH}^n(A)$ is a commutative graded algebra. The Yoneda product on $\mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(A, A)$ coincides with the cup product on the Hochschild cohomology $\mathrm{HH}^*(A)$.

Following [1], we recall the Lie bracket $[\ ,\ ]$ on Hochschild cohomology ring $\mathrm{HH}^*(A)$. First, we recall the definition of Gerstenhaber algebras.

For $f \in C^n(A)$ and $g \in C^m(A)$ $(n + m \geq 1)$, we define $[f, g] \in C^{n+m-1}(A)$ as follows: If $n, m \geq 1$, then for $1 \leq i \leq n$, $f \circ_i g \in C^{n+m-1}(A)$ is given by

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}).$$

If $n \geq 1$ and $m = 0$, then for $1 \leq i \leq n$ $f \circ_i g \in C^{n-1}(A)$ is given by

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g \otimes a_i \otimes \cdots \otimes a_{n-1})$$

where g is regarded as an element of A.

We set

$$f \circ g := \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} f \circ_i g$$

and

$$[f, g] := f \circ g - (-1)^{(n-1)(m-1)} g \circ f \in C^{n+m-1}(A).$$

Then $[\ ,\ ]$ induces $[\ ,\ ] : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \to \mathrm{HH}^{n+m-1}(A)$. Then $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$ is a Gerstenhaber algebra.

Following [4], we recall BV differentials on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$ focusing on Frobenius algebras. Let A be a Frobenius algebra with the bilinear form $\langle , \rangle$ and the Nakayama automorphism ν. The map $\phi_\nu : C^n(A) \to C^n(A)$ can be defined by

$$(\phi_\nu(f))(a_1 \otimes \cdots \otimes a_n) = \nu^{-1}(f(\nu(a_1) \otimes \cdots \otimes \nu(a_n)))$$

for $f \in C^n(A)$ and $a_i \in A$. Then $\phi_\nu(\delta_n f) = \delta_n(\phi_\nu(f))$, so ϕ_ν induces an automorphism of the Hochschild cohomology. Let $C^n(A)^\nu = \{f \in C^n(A) \mid \phi_\nu(f) = f\}$. Then, δ_n restricts to a differential $\delta^\nu_n : C^n(A)^\nu \to C^{n-1}(A)^\nu$ and let $\mathrm{HH}^n(A)^{\nu\uparrow}$ be the n-th cohomology of the complex $(C^*(A)^\nu, \delta^\nu_*)$. Then $\mathrm{HH}^n(A)^{\nu\uparrow} \cong \mathrm{HH}^n(A)$ if ν is diagonalizable by [4, Corollary 2]. The cup product on the Hochschild complex can restrict to $(C^*(A)^\nu, \delta^\nu_*)$ and $\mathrm{HH}^*(A)^{\nu\uparrow}$ has a ring structure. Gerstenhaber algebra structure on $\mathrm{HH}^*(A)$ induces Gerstenhaber algebra structure on $\mathrm{HH}^*(A)^{\nu\uparrow}$.

Let $f \in C^n(A)$, $n \geq 1$. Then for $i (1 \leq i \leq n)$ we define $\Delta_i f \in C^{n-1}(A)$ by

$$\langle \Delta_i f(a_1 \otimes \cdots \otimes a_{n-1}), a_n \rangle = \langle f(a_i \otimes \cdots \otimes a_n \otimes \nu a_1 \otimes \cdots \otimes \nu a_{i-1}), 1 \rangle$$

and we set $\Delta := \sum_{i=1}^{n} (-1)^{i(n-1)} \Delta_i : C^n(A) \to C^{n-1}(A)$.

Theorem 4 ([4, Theorem 2.2]). *Let A be a Frobenius algebra with the bilinear form $\langle\ ,\ \rangle$ and the Nakayama automorphism ν. Then, Δ as above induces a BV algebra structure on the Gerstenhaber algebra* $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$.

3. Hochschild cohomology of self-injective Nakayama algebras

Let $\Lambda = KZ_e/J^N$ be a self-injective Nakayama algebra with the Nakayama automorphism ν, where Z_e is a cyclic quiver with $e(\geq 2)$ vertices, J is the radical of KZ_e and $N \geq 2$. Moreover, let $g_0 = \gcd(N-1, e)$. Then, $\mathrm{ord}(\nu) = \frac{e}{g_0}$.

In this section, we give the cohomology ring $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ and BV differential Δ on $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ defined by Volkov [4] of the case $\mathrm{char}\, K \mid \mathrm{ord}(\nu) = \frac{e}{g_0}$. In this case, $N \not\equiv 1 \pmod{e}$ and the Nakayama automorphism ν is not necessarily diagonalizable.

First, we give the cohomology ring $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$.

Theorem 5. *Suppose that $N \leq e$ and either $\mathrm{char}\, K \nmid N$ or $\mathrm{char}\, K \mid N$ and $\gcd(N, e) \neq 1$. For $0 \leq j \leq N-2$, let $i_j(>0)$ be the smallest integer such that $Ni_j \equiv j \pmod{e}$ if there exists an integer k such that $Nk \equiv j \pmod{e}$. Then $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ is isomorphic to*

$$K[y, z_j \mid 0 \leq j \leq N-2 \text{ and there exists an integer } k \text{ such that } Nk \equiv j \,(\mathrm{mod}\, e)]/I,$$

where $\deg y = 1$, $\deg z_j = 2i_j$ and the ideal I is given by the following relations:

$$y^2 = 0,\ z_j^{[\frac{N}{j}]} = 0,$$
$$z_p z_q = z_{p+q} \text{ if } 2 \leq p+q \leq N-2 \text{ and } i_{p+q} = i_p + i_q \text{ for } 1 \leq p, q \leq N-2$$

Theorem 6. *Suppose that $N > e$ and either $\mathrm{char}\, K \nmid N$, or $\mathrm{char}\, K \mid N$ and $\gcd(N, e) \neq 1$. For $0 \leq r \leq e-1$, let $i_r(>0)$ be the smallest integer such that $Ni_r \equiv r \pmod{e}$ if there exists an integer k such that $Nk \equiv r \pmod{e}$. Then $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ is isomorphic to*

$$K[x, y, z_r \mid 0 \leq r \leq e-1 \text{ and there exists an integer } k \text{ such that } Nk \equiv r \,(\mathrm{mod}\, e)]/I,$$

where $\deg x = 0$, $\deg y = 1$, $\deg z_r = 2i_r$ and the ideal I is given by the following relation:

$$x^m = 0 \text{ if } N \equiv 0 \,(\mathrm{mod}\, e),\ x^{m+1} = 0 \text{ if } N \not\equiv 0 \,(\mathrm{mod}\, e),\ y^2 = 0,\ z_r^{[\frac{N}{r}]} = 0,$$

$$z_p z_q = \begin{cases} z_{p+q}x^k & \text{if } m = 1,\ 1 \leq p+q \leq N-2,\ p+q = ke+l\ (0 \leq l \leq t) \\ & \text{and } i_p + i_q = i_{p+q} \\ z_p z_q = z_l x^k & \text{if } m > 1,\ 1 \leq p+q \leq N-2 \text{ and } p+q = ke+l\ (0 \leq l \leq e-1) \\ 0 & \text{otherwise} \end{cases}$$

Corollary 7. *For self-injective Nakayama algebra Λ, $\mathrm{HH}^*(\Lambda) \cong \mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ as graded commutative algebras holds.*

Next, we give the BV differentials on $(\mathrm{HH}^*(\Lambda)^{\nu\uparrow}, \smile, [\ ,\ ])$.

Theorem 8. *Suppose that $N \leq e$ and either $\mathrm{char}\, K \nmid N$, or $\mathrm{char}\, K \mid N$ and $\gcd(N, e) \neq 1$. Then, the BV differential $\Delta : \mathrm{HH}^*(\Lambda)^{\nu\uparrow} \to \mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ is given by*

$$\Delta(1) = \Delta(z_j) = \Delta(z_j z_{j'}) = 0,$$
$$\Delta(y) = \frac{N-1}{g_0}, \Delta(yz_j) = \frac{N-1}{g_0} z_j$$

for generators of $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ and their multiplications. Moreover, $[\ ,\ ] = 0$.

Theorem 9. *Suppose that $N > e$ and either* $\operatorname{char} K \nmid N$, *or* $\operatorname{char} K \mid N$ *and* $\gcd(N, e) \neq 1$. *Then, BV differential* $\Delta : \mathrm{HH}^*(\Lambda)^{\nu\uparrow} \to \mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ *is given by*

$$\Delta(1) = \Delta(x) = \Delta(x^2) = \Delta(z_r) = \Delta(z_r z_{r'}) = 0,$$
$$\Delta(y) = \frac{N-1}{g_0}, \Delta(xy) = \frac{N-1}{g_0}x, \Delta(yz_r) = \frac{N-1}{g_0}z_r$$

for generators of $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ *and their multiplications. Moreover,* $[\ ,\ ] = 0$.

Finally, we obtain the non trivial Batalin Vilkovisky algebra structure for the self-injective Nakayama algebra in [2, Example 5.3].

Example 10 (cf. [2, Example 5.3])**.** Suppose that $\operatorname{char} K = 2$, $e = 2$ and $N = 4$. Then, $g_0 = 1$, $\operatorname{ord}(\nu) = 2$ and $\mathrm{HH}^*(\Lambda)^{\nu\uparrow} = K[x, y, z_0]/(x^2, y^2)$, where $\deg x = 0$, $\deg y = 1$ and $\deg z_0 = 2$. Moreover, the bracket $[\ ,\ ] = 0$ and BV differential Δ is given by

$$\Delta(1) = \Delta(x) = \Delta(z_0) = \Delta(xz_0) = \Delta(z_0^2) = 0,$$
$$\Delta(y) = 1, \Delta(yx) = x, \Delta(yz_0) = z_0.$$

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